3. VOL'MIR A.S., Stability of Deformable Systems. Nauka, Moscow, 1967.
4. TIMOSHENKO S.P., Stability of Elastic Systems. Gostekhizdat, Moscow, 1955.
5. FEODOS'EV V.I., Strength of Materials. Nauka, Moscow, 1970.
6. MIKHLIN S.G., Linear Partial Differential Equations. Vysh. Shkola, Moscow, 1977.
7. ARUTYUNYAN N.KH., Some Questions of Creep Theory. Gostekhizdat, Moscow-Leningrad, 1952.
8. LADYZHENSKAYA O.A., Boundary Value Problems of Mathematical Physics, Nauka, Moscow, 1973.
9. MARKUSHIN V.M., Critical values of the radial compressive force intensity for thin circular plates. Raschety na Prochnost', 4, 1959; 5, 1960; 6, 1960.

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A MATHEMATICAL MODEL OF THE PROBLEM OF DIAGNOSING A THERMOELASTIC MEDIUM*

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The diagnosis problem is understood to mean the problem of determining material characteristics by means of information on the physical fields originating therein under the influence of external effects. The problem is from the class of inverse problems of mathematical physics /1/ and is solved using the model of generalized thermomechanics for weakly anisotropic media. As a result of the analysis of wave processes in such a medium, a method is developed for determining the thermoelastic characteristics by means of the temperature and displacement values on the half-space boundary. Examples of calculating specific characteristics are examined.

1. We shall consider the problem of diagnosing a thermoelastic medium within the framework of the model of generalized thermomechanics /2/

$$q_{j,j} + C_\varepsilon \Theta^{\cdot} + T_0 \beta_{ij} \varepsilon_{ij}^{\cdot} = 0, \quad \tau q_j^{\cdot} + q_j = -K_{ij}\Theta_{,i}, \quad \sigma_{ij,j} = \rho u_i^{\cdot\cdot} \tag{1.1}$$
$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\Theta, \quad i, j, k, l = 1, 2, 3$$

Here C_ε is the specific heat for constant deformation, $\beta_{ij} = C_{ijkl}\alpha_{kl}$, α_{kl} are the coefficients of linear expansion, C_{ijkl} are the isothermal stiffness coefficients of an anisotropic material, K_{ij} are the thermal conductivities, τ is the heat flux relaxation time, ρ is the density (the quantities listed above are functions of the space variables $\mathbf{x} = (x_1, x_2, x_3)$), q_j are components of the heat flux vector, $\Theta = (T - T_0)$ is the relative body temperature, $\varepsilon_{ij}, \sigma_{ij}$ are the strain and stress tensors, u_i are the displacement components (these quantities are functions of $\mathbf{x}$ and the time t), and $T_0 = \mathrm{const}$ is the body temperature in the natural state. The dots denote partial derivatives with respect to time, the subscript after the comma is the derivative with respect to the corresponding coordinate. Summation is over repeated subscripts.

Unlike the dynamic equations of the theory of elasticity and the non-stationary heat conduction equations, the generalized thermomechanics equations describe the mutual influence of the deformation and temperature fields and also take account of the finiteness of the heat propagation velocity. It is important to take these effects into account in any study of the qualitative behaviour of the solution. At the same time, in quantitative respects taking them into account does not result in any appreciable difference between the solutions and the solutions of the elasticity and heat conduction theories in many cases /2, 3/.

In view of this, we will assume that the terms $T_0\beta_{ij}\varepsilon_{ij}^{\cdot}$, $\tau q_j^{\cdot}$, $\beta_{ij}\Theta$ are small quantities of the order of ε $(0 < \varepsilon \ll 1)$ and the solution of system (1.1) $\{q_j, \Theta, \sigma_{ij}, \varepsilon_{ij}, u_i\}$ differs from the function $\{q_j^\circ, \Theta^\circ, \sigma_{ij}^\circ, u_i^\circ\}$, which is a solution of the mutually uncoupled non-stationary equations of the theory of heat conduction and the dynamic equations of elasticity theory, by a quantity $O(\varepsilon)$

$$\mathring{q}_{j,j} + C_\varepsilon^\circ \Theta^{\circ\cdot} = 0, \quad q_j^\circ + K_{ij}^\circ \Theta_{,i}^\circ = 0, \quad \mathring{\sigma}_{ij,j} = \rho^\circ \mathring{u}_i^{\cdot\cdot} \tag{1.2}$$
$$\varepsilon_{ij}^\circ = \frac{1}{2}(\mathring{u}_{i,j} + \mathring{u}_{j,i}), \quad \sigma_{ij}^\circ = \mathring{C}_{ijkl}\mathring{\varepsilon}_{kl}, \quad i, j, k, l = 1, 2, 3$$

Here we assume that $|C_\varepsilon - C_\varepsilon^\circ|$, $|K_{ij} - K_{ij}^\circ|$, $|\rho - \rho^\circ|$, $|C_{ijkl} - C_{ijkl}^\circ|$ are also of the order

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of ε, while the thermoelastic characteristics $C_\varepsilon^\circ, K_{ij}^\circ, \rho^\circ, C_{ijkl}^\circ$ correspond to a homogeneous isotropic medium. This last assumption denotes weak inhomogeneity and weak anisotropy of the thermoelastic medium under investigation. The slight difference between the material properties and the properties of a homogeneous isotropic thermoelastic medium can be caused by an insignificant disturbance of technology during production or be the result of the influence of external effects when using the piece.

In general, even a small change in the magnitude of the material characteristics (the coefficients of (1.1)) for invariant initial and boundary conditions can result in a substantial change in the nature of the dynamic process (i.e., the solutions of (1.1)). To avert such situations, it is assumed in the theory of inverse problems that the assertion of a small change in the solution for a small change in the coefficients /4/ holds for the class of solutions under consideration and for the set of variations of the coefficients of the equations. We note that this assumption is known to be satisfied in investigations of the solutions in small time intervals, not least because of the continuous dependence of the solution on the initial conditions, which remain invariant. We will later assume that $q_i^\varepsilon = q_i - q_i^\circ$, $\Theta^\varepsilon = \Theta - \Theta^\circ$, $\sigma_{ij}^\varepsilon = \sigma_{ij} - \sigma_{ij}^\circ$, $\varepsilon_{ij}^\varepsilon = \varepsilon_{ij} - \varepsilon_{ij}^\circ$, $u_i^\circ = u_i - u_i^\circ$ are small quantities of the order of ε.

Assuming the quantities in (1.2) as well as the requisite number of their derivatives to be of the order of unity, and discarding terms of an order of smallness less than ε, we reduce (1.1) to the form

$$C_\varepsilon^\circ \Theta^{\varepsilon\cdot} - K_{ij}^\circ \overset{\varepsilon}{\Theta}_{,ij} = -C_\varepsilon^\varepsilon \Theta^{\circ\cdot} + (K_{ij}^\varepsilon \Theta_{,i}^\circ)_{,j} - T_0 \beta_{ij} \overset{\circ\cdot}{u}_{i,j} - (\tau K_{ij}^\circ \Theta_{,i}^\circ)_{,j} \tag{1.3}$$
$$\rho^\circ \overset{\varepsilon\cdot\cdot}{u_i} - \overset{\circ}{C}_{ijkl} \overset{\varepsilon}{u}_{k,lj} = -\rho^\varepsilon \overset{\circ\cdot\cdot}{u_i} + (\overset{\varepsilon}{C}_{ijkl} \overset{\circ}{u}_{k,l})_{,j} + (\beta_{ij}\Theta^\circ)_{,j}$$

We reduce (1.2) to the form

$$C_\varepsilon^\circ \Theta^{\circ\cdot} - K_{ij}^\circ \overset{\circ}{\Theta}_{,ij} = 0, \quad \rho^\circ \overset{\circ\cdot\cdot}{u_i} - \overset{\circ}{C}_{ijkl} \overset{\circ}{u}_{k,lj} = 0 \tag{1.4}$$
$$C_\varepsilon^\varepsilon = C_\varepsilon - C_\varepsilon^\circ, \quad K_{ij}^\varepsilon = K_{ij} - K_{ij}^\circ, \quad \rho^\varepsilon = \rho - \rho^\circ, \quad \overset{\varepsilon}{C}_{ijkl} = C_{ijkl} - \overset{\circ}{C}_{ijkl}$$

where by virtue of the isotropy of the basic medium, the tensor quantities K_{ij}°, $\overset{\circ}{C}_{ijkl}$ have the form /5/

$$K_{ij}^\circ = \delta_{ij} K^\circ, \quad i, j = 1, 2, 3; \quad \overset{\circ}{C}_{1111} = \overset{\circ}{C}_{2222} = \overset{\circ}{C}_{3333} = \lambda^\circ + 2\mu^\circ$$
$$\overset{\circ}{C}_{1122} = \overset{\circ}{C}_{1133} = \overset{\circ}{C}_{2233} = \lambda^\circ, \quad \overset{\circ}{C}_{1212} = \overset{\circ}{C}_{1313} = \overset{\circ}{C}_{2323} = 2\mu^\circ$$
$$\overset{\circ}{C}_{1112} = \overset{\circ}{C}_{1113} = \overset{\circ}{C}_{1123} = \overset{\circ}{C}_{1323} = \overset{\circ}{C}_{1213} = \overset{\circ}{C}_{2213} = \overset{\circ}{C}_{2212} = \overset{\circ}{C}_{2223} =$$
$$\overset{\circ}{C}_{1223} = \overset{\circ}{C}_{3313} = \overset{\circ}{C}_{3312} = \overset{\circ}{C}_{3323} = 0$$

where λ°, μ° are Lamé constants, and δ_{ij} is the Kronecker delta.

We will examine thermoelastic wave propagation in the half-space $x_3 \geqslant 0$. Eqs.(1.3) and (1.4) are closed by the initial and boundary conditions

$$\Theta^\circ(\mathbf{x}, 0) = \varphi_1(\mathbf{x}), \quad \mathbf{u}^\circ_{i}(\mathbf{x}, 0) = \varphi_2(\mathbf{x}), \quad \mathbf{u}^{\circ\cdot}(\mathbf{x}, 0) = \varphi_3(\mathbf{x}) \tag{1.5}$$
$$\Theta^\varepsilon(\mathbf{x}, 0) = 0, \quad \mathbf{u}^\varepsilon(\mathbf{x}, 0) = 0, \quad \mathbf{u}^{\varepsilon\cdot}(\mathbf{x}, 0) = 0$$
$$\Theta_{,3}^\circ(x_1, x_2, 0, t) = \psi_1(x_1, x_2, t), \quad \mathbf{u}_{,3}^\circ(x_1, x_2, 0, t) = \psi_2(x_1, x_2, t)$$
$$\Theta_{,3}^\varepsilon(x_1, x_2, 0, t) = 0, \quad \mathbf{u}_{,3}^\varepsilon(x_1, x_2, 0, t) = 0$$

Therefore, the initial and boundary conditions in the problem of thermoelastic wave propagation in a weakly anisotropic, weakly inhomogeneous medium, taking the connectedness of the deformation and temperature fields and the finiteness of the heat propagation velocity into account (in linearized form), will agree with the corresponding conditions for the basic dynamic process being considered without taking account of these effects in a basic homogeneous isotropic medium.

We will consider additional data on the boundary of the thermoelastic half-space

$$\Theta^\varepsilon(x_1, x_2, 0, t) = \chi_1(x_1, x_2, t), \ \operatorname{div} \mathbf{u}^\varepsilon(x_1, x_2, 0, t) = \chi_2(x_1, x_2, t) \tag{1.6}$$
$$\operatorname{rot} \mathbf{u}^\varepsilon(x_1, x_2, 0, t) = \chi_3(x_1, x_2, t)$$

as information obtained as a result of test experiments.

This information will be the basis for determining the unknown thermoelastic characteristics of the material $C_\varepsilon^\varepsilon, K_{ij}^\varepsilon, \rho^\varepsilon, \overset{\varepsilon}{C}_{ijkl}, \beta_{ij}, \tau$, which are considered to be sufficiently smooth functions of the space variables. It is conceivable that several experiments will be required to determine all these characteristics. We shall ascribe the superscript n to quantities corresponding to the n-th experiment when it is necessary to emphasize this.

We note that within the framework of the diagnosis problem, Eqs.(1.1) are non-linear since they contain products of the desired thermoelastic characteristics of the medium by the also unknown functions $q_i, \Theta, \sigma_{ij}, \varepsilon_{ij}, u_i$ describing the dynamic process in this medium. The passage to (1.3) and (1.4) is essentially a linearization in the small parameter ε.

Further solution of the problem will result in the appearance of differential equations in the unknown thermoelastic characteristics. Boundary conditions in $\{\tau(\mathbf{x}), \rho^\varepsilon(\mathbf{x}), C_\varepsilon^\varepsilon(\mathbf{x}), \beta_{ij}(\mathbf{x}), K_{ij}^\varepsilon(\mathbf{x}), C_{ijkl}^\varepsilon(\mathbf{x})\}$ for $x_3 = 0$ are necessary for their solutions; we will consider them to be homogeneous without loss of generality.

2. That regime of force and temperature loading is naturally utilized in carrying out the experiments for which the dynamic process in the basic homogeneous isotropic medium will be sufficiently simple in nature. For simplicity we shall consider just such loadings as will cause dynamic processes of the form

$$\Theta^\circ = p(\mathbf{x}) \exp(-at), \quad \mathbf{u}^0 = \mathbf{g}(\mathbf{x}) \exp(-at)$$
$$a = \text{const}, \quad a > 0, \quad \mathbf{g}(\mathbf{x}) = \mathbf{g}_{(1)}(\mathbf{x}) + \mathbf{g}_{(2)}(\mathbf{x})$$

which will damp out exponentially with time in the basis medium, where $\operatorname{rot} \mathbf{g}_{(1)} = 0$, $\operatorname{div} \mathbf{g}_{(2)} = 0$.

It follows from (1.4) that the functions p and g should satisfy equations of the type of the Helmholtz Eqs.(6)

$$a^2 \mathbf{g}_{(1)} - c_1^2 \Delta \mathbf{g}_{(1)} = 0, \quad a^2 \mathbf{g}_{(2)} - c_2^2 \Delta \mathbf{g}_{(2)} = 0$$
$$C_\varepsilon^\circ ap + K^\circ \Delta p = 0, \quad c_1 = \sqrt{(\lambda^\circ + 2\mu^\circ)/\rho^\circ}, \ c_2 = \sqrt{\mu^\circ/\rho^\circ}$$

Methods of solving such equations are known /6/; consequently, we shall henceforth consider the functions p and g as known.

Applying the operators div and rot to the second equation in (1.3), we reduce system (1.3) to a system of equations of the form

$$\operatorname{div} \mathbf{u}^{\varepsilon\cdot\cdot} - c_1^2 \Delta (\operatorname{div} \mathbf{u}^\varepsilon) = \operatorname{div} \mathbf{F}, \quad \mathbf{u}^\varepsilon = (u_1^\varepsilon, \ u_2^\varepsilon, \ u_3^\varepsilon) \tag{2.1}$$
$$\operatorname{rot} \mathbf{u}^{\varepsilon\cdot\cdot} - c_2^2 \Delta (\operatorname{rot} \mathbf{u}^\varepsilon) = \operatorname{rot} \mathbf{F}, \ \mathbf{F} = (F_1, \ F_2, \ F_3) \tag{2.2}$$
$$C_\varepsilon^\circ \Theta^{\varepsilon\cdot} - K^0 \Delta \Theta^\varepsilon = \Phi \tag{2.3}$$
$$F_i = -\frac{\exp(-at)}{\rho^\circ} \left[a^2 \rho^\varepsilon g_i - (C_{ijkl}^\varepsilon g_{k,l})_{,j} + (\beta_{ij} p)_{,j}\right]$$
$$\Phi = \exp(-at) [C_\varepsilon^\varepsilon \ ap + (K_{ij}^\varepsilon p_{,j})_{,i} + aT_0 \beta_{ij} g_{i,j} + aK^\circ \ (\tau p_{,i})_{,i}$$

We apply the operator $\partial/\partial t + aI$ to (2.1)-(2.3), where I is the unit operator. We obtain

$$v^{\cdot\cdot} - c_1^2 \Delta v = 0 \tag{2.4}$$
$$\boldsymbol{\omega}^{\cdot\cdot} - c_2^2 \Delta \boldsymbol{\omega} = 0, \quad \boldsymbol{\omega} = (\omega_1, \ \omega_2, \ \omega_3) \tag{2.5}$$
$$C_\varepsilon^\circ T^\cdot - K^\circ \Delta T = 0 \tag{2.6}$$
$$(v = \operatorname{div} \mathbf{u}^{\varepsilon\cdot} + a \operatorname{div} \mathbf{u}^\varepsilon, \ \boldsymbol{\omega} = \operatorname{rot} \mathbf{u}^{\varepsilon\cdot} + a \operatorname{rot} \mathbf{u}^\varepsilon, \ T = \Theta^{\varepsilon\cdot} + a\Theta^\varepsilon)$$

We have initial and boundary conditions for (2.4)-(2.6)

$$v(\mathbf{x}, \ 0) = 0 \tag{2.7}$$
$$\dot{\boldsymbol{\omega}}(\mathbf{x}, \ 0) = 0 \tag{2.8}$$
$$v(x_1, \ x_2, \ 0, \ t) = h_1(x_1, \ x_2, \ t), \ v_{,3}(x_1, \ x_2, \ 0, \ t) = h_2(x_1, \ x_2, \ t) \tag{2.9}$$
$$\boldsymbol{\omega}(x_1, \ x_2, \ 0, \ t) = \mathbf{h}_3(x_1, \ x_2, \ t), \ \boldsymbol{\omega}_{,3}(x_1, \ x_2, \ 0, \ t) = \mathbf{h}_4(x_1, \ x_2, \ t) \tag{2.10}$$
$$\mathbf{h}_3 = (h_{31}, \ h_{32}, \ h_{33}), \ \mathbf{h}_4 = (h_{41}, \ h_{42}, \ h_{43})$$
$$T(x_1, \ x_2, \ 0, \ t) = \chi_1^\cdot + a\chi_1, \quad T_{,3}(x_1, \ x_2, \ 0, \ t) = 0 \tag{2.11}$$

where the functions $h_1 - \mathbf{h}_4$ are expressed in terms of χ_2, χ_3 and are henceforth considered known.

Eqs.(2.4), (2.5) with the initial conditions (2.7), (2.8) and the boundary conditions (2.9), (2.10) are non-hyperbolic Cauchy problems for the wave equations that have been examined in detail in /7/, say.

We consider problem (2.4), (2.7), (2.9). We represent the function $v(\mathbf{x}, t)$ in the form $v(\mathbf{x}, t) = v_1(\mathbf{x}, t) + v_2(\mathbf{x}, t)$, where we have for v_1

$$v_1^{\cdot\cdot} - c_1^2 \Delta v_1 = 0 \tag{2.12}$$
$$v_1(\mathbf{x}, 0) = 0, \ v_1(x_1, x_2, 0, t) = h_1(x_1, x_2, t), \ v_{1,3}(x_1, x_2, \ 0, t) = 0 \tag{2.13}$$

and the function v_2 is such that

$$v_2^{\cdot\cdot} - c_1^2 \Delta v_2 = 0 \tag{2.14}$$
$$v_2(\mathbf{x}, 0) = 0, \ v_2(x_1, \ x_2, 0, t) = 0, \ v_{2,3}(x_1, \ x_2, 0, t) = h_2(x_1, x_2, t) \tag{2.15}$$

An explicit formula enabling $v^\cdot(\mathbf{x}, 0)$ to be determined in a problem of the form (2.12), (2.13) by means of the functions $h_1(x_1, x_2, t)$, i.e., the reduction of problem (2.12), (2.13) to an ordinary Cauchy problem is presented in /8/. The functions v_2, ω are determined analogously.

Problem (2.6), (2.11) can also be reduced to a non-hyperbolic Cauchy problem for the wave equation. According to /9/

$$T(\mathbf{x}, t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\eta^2}{4t}\right) U(\mathbf{x}, \eta)\, d\eta$$

where $U(\mathbf{x}, t)$ is the solution of the Cauchy non-hyperbolic problem

$$C_\varepsilon^\circ U^{\cdot\cdot} - K^\circ \Delta U = 0, \; -\infty < x_1, x_2 < \infty, \; 0 \leqslant x_3, t < \infty$$

$$U(\mathbf{x}, 0) = 0, \; U(x_1, x_2, 0, t) = \chi(x_1, x_2, t), \; U_{,3}(x_1, x_2, 0, t) = 0$$

The function χ is found by inverting the relation

$$\chi_1^{\cdot} + a\chi_1 = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\eta^2}{4t}\right) \chi(x_1, x_2, \eta)\, d\eta$$

The invertibility of relationships of this form is shown in /10/.

Therefore, the functions $v^{\cdot}(\mathbf{x}, 0)$, $\boldsymbol{\omega}^{\cdot}(x, 0)$, $T(\mathbf{x}, 0)$ can henceforth be considered known. This means that $\operatorname{div} \mathbf{u}^{\varepsilon\cdot\cdot}(\mathbf{x}, 0) = v^{\cdot}(\mathbf{x}, 0)$, $\operatorname{rot} \mathbf{u}^{\varepsilon\cdot\cdot}(\mathbf{x}, 0) = \boldsymbol{\omega}^{\cdot}(\mathbf{x}, 0)$, $\Theta^{\varepsilon\cdot}(\mathbf{x}, 0) = T(\mathbf{x}, 0)$, are known, and of course, the functions $\operatorname{div} F(\mathbf{x}, 0), \operatorname{rot} \mathbf{F}(\mathbf{x}, 0), \Phi(\mathbf{x}, 0)$, obtained on substituting the values $\operatorname{div} \mathbf{u}^{\varepsilon\cdot\cdot}(\mathbf{x}, 0), \operatorname{rot} \mathbf{u}^{\varepsilon\cdot\cdot}(\mathbf{x}, 0), \operatorname{div} \mathbf{u}^{\varepsilon\cdot}(\mathbf{x}, 0), \operatorname{rot} \mathbf{u}^{\varepsilon\cdot}(\mathbf{x}, 0), \Theta^{\varepsilon\cdot}(\mathbf{x}, 0), \Theta^{\varepsilon}(\mathbf{x}, 0)$ into (2.1)-(2.3) are also known.

Since $\mathbf{F}(x_1, x_2, 0, 0) = 0$ by virtue of the boundary conditions taken for the values of the desired thermoelastic characteristics, then the function $\mathbf{F}(\mathbf{x}, 0)$ can be restored by means of the values of $\operatorname{div} \mathbf{F}(\mathbf{x}, 0), \operatorname{rot} \mathbf{F}(\mathbf{x}, 0)$. We note that for the restoration of three components of the vector-function $\mathbf{F}(\mathbf{x}, 0)$ a knowledge of the values of $\operatorname{div} \mathbf{F}(\mathbf{x}, 0)$ and of only two components of the vector function $\operatorname{rot} \mathbf{F}(\mathbf{x}, 0)$ is sufficient. Therefore, as information to be utilized to solve the problem it is sufficient to take just two components of the vector-function $\boldsymbol{\chi}_3$. This is, however, already seen from the very form of the information (1.6), where the third component $\boldsymbol{\chi}_3$ is not independent of the first two components and of χ_2.

We also note that for the uniqueness of the solution of the non-hyperbolic Cauchy problem substantially utilized here for the wave equation, it is sufficient to give the boundary value of the desired function not in the whole $x_3 = 0$ plane but only in the domain $\{x_1^2 + x_2^2 \leqslant r, x_3 = 0\}$, where r i an arbitrary fixed positive number /11/, i.e., information for the diagnosis problem could also be given just in this domain. The need to consider information in the whole $x_3 = 0$ plane is associated with utilization of the explicit representation from /8/. We note that the classical ill-posed nature of the diagnosis problem under cosideration follows from the classical ill-posed nature of the non-hyperbolic Cauchy problem.

Now, using the definitions of the functions Φ and $\mathbf{F}$, we can write the following equations permitting the determination of the desired characteristics of a thermoelastic medium

$$C_\varepsilon^\varepsilon\, ap + (K_{ij}^\varepsilon p_{,i})_{,j} + aT_0 \beta_{ij} g_{i,j} + aK^\circ (\tau p_{,i})_{,i} = \Phi(\mathbf{x}, 0) \tag{2.16}$$

$$1/\rho^\circ [a^2 \rho^\varepsilon g_i - (C_{ijkl}^\varepsilon g_{k,l})_{,j} + (\beta_{ij} p)_{,j}] = -F_i(\mathbf{x}, 0), \tag{2.17}$$

$$i, j, k, l = 1, 2, 3$$

System (2.16)-(2.17) is obtained when utilizing one test experiment with the measurement of four physical quantities $\operatorname{div} \mathbf{u}$, $(\operatorname{rot} \mathbf{u})_1$, $(\operatorname{rot} \mathbf{u}_2)$, Θ on the half-space boundary and consists of four equations. In the general case the number of desired thermoelastic characteristics of the medium is 36, consequently, nine independent tests are sufficient for their determination. The functions $\mathbf{g}^{(n)}(x)$, $p^{(n)}(x)$ governing the basic dynamic process for the n-th experiment should be independent for different $n, n = 1, 2, \ldots, 9$.

We note that the unknown characteristics $C_\varepsilon^\varepsilon, K_{ij}^\varepsilon, \tau$ can be determined only from equations of the form (2.16), the characteristics ρ^ε, C_{ijkl}^ε only from equations of the form (2.17) while β_{ij} can be determined both from those and from other equations since they are in both (2.16) and (2.17). However, equations of the form (2.16) are algebraic in β_{ij} while terms of the form $\beta_{ij,j}$ are in (2.17). Consequently, β_{ij} are conveniently determined from equations of the form (2.16), which results in the need to carry out fourteen instead of nine independent experiments with a measurement of the temperature on the half-space boundary. The number of deformation measurements on the half-space boundary can here be cut down correspondingly from 27 to 22.

The boundary conditions

$$\tau(x_1, x_2, 0) = K_{ij}^\varepsilon (x_1, x_2, 0) = C_{ijkl}^\varepsilon (x_1, x_2, 0) = 0,$$
$$i, j, k, l = 1, 2, 3$$

that correspond to agreement between the mentioned thermoelastic characteristics of the investigated and basic media on the half-space boundary, can be used to close the system of equations of the form (2.16) and (2.17).

The determination of certain thermoelastic characteristics from equations of the form (2.16) and (2.17) can be simplified substantially by a special choice of the functions $g^{(n)}(x)$ and $p^{(n)}(x)$, i.e. by a special selection of the nature of the force and temperature loading during the test experiments.

We illustrate this by the example of the determination of the unknown thermal volume expansion coefficients β_{ij}. We will consider six test experiments characterising just the force loading, i.e., $p^{(n)}(x) = 0$, $g_i^{(n)}(x) = G_i^{(n)}(x)$, $i = 1, 2, 3$, $n = 1, 2, \ldots, 6$. In making these experiments we limit ourselves to temperature measurement on the half-space boundary. We then obtain from equations of the form (2.16)

$$G_{i,j}^{(n)}\beta_{ij} = \frac{1}{T_0 a^{(n)}}\,\Phi(\mathbf{x}, 0);\quad i, j = 1, 2, 3;\quad n = 1, 2, \ldots, 6$$

which means that six independent components of the tensor β_{ij} are determined from a system of six linear algebraic equations. The sufficient condition for this system ot be solvable will be

$$\det(A_{kl}) \neq 0,\quad A_{kl} = G_{i,j}^{(k)} + (1-\delta_{ij})\,G_{j,i}^{(k)};\quad k, l = 1, 2, \ldots, 6$$
$$(i, j) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$$

An assumption about the possibility of conducting a sufficiently large number of test experiments simplifies the problem still more. Then the partial derivatives of the unknown thermoelastic characteristics present in equations of the form (2.16) and (2.17) can be considered as separate desired functions. Therefore, the number of desired functions is increased substantially, but then these unknown functions are related by an linear system of algebraic equations. Such a formulation of the problem requires no boundary conditions in the unknown thermoelastic characteristics of the medium, however, after the problem has been solved it is necessary to verify whether some of the functions found are actually partial derivatives of others as had been assumed. In addition, confirmation of the correspondence between the thermoelastic characteristics of the material obtained during solution of the diagnosis problem and their physical meaning is necessary.

3. We will consider a case in detail, when the external effect changing the thermoelastic characteristics of an initially homogeneous and isotropic material is such that the change in these characteristics depends only on the distance to the half-space boundary, i.e., on the coordinate x_3. We also assume that the external effect under consideration would result in weak inhomogeneity of the material while conserving its isotropy.

In this case it is best to examine one-dimensional dynamic processes (dependent only on the time t and one space coordinate x_3). Eqs.(1.3) describing the dynamic processes in the material in the general case are now simplified

$$C_\varepsilon^\circ\Theta^{\varepsilon\cdot} - K^\circ\Theta^\varepsilon_{,33} = -C_\varepsilon^\varepsilon\Theta^{\circ\cdot} + (K^\varepsilon\Theta_{,3}^\circ)_{,3} - T_0\beta\mathring{u}_{3,3}^{\cdot} - K^\circ(\tau\Theta_{,3}^{\circ\cdot})_{,3} \tag{3.1}$$

$$\rho^\circ u_i^{\varepsilon\cdot\cdot} - \mu^\circ u^\varepsilon_{i,33} = -\rho^\varepsilon \mathring{u}_i^{\cdot\cdot} + (\mu^\varepsilon \mathring{u}_{i,3})_{,3},\quad i = 1, 2 \tag{3.2}$$

$$\rho^\circ u_3^{\varepsilon\cdot\cdot} - E^\circ u^\varepsilon_{3,33} = -\rho^\varepsilon u^{\circ\cdot\cdot} + (E^\varepsilon \mathring{u}_{3,3})_{,3} + (\beta\Theta^\circ)_{,3} \tag{3.3}$$

while the number of desired thermoelastic characteristics is reduced to seven $C_\varepsilon^\varepsilon$, K^ε, τ, β, ρ^ε, μ^ε, $E^\varepsilon = \lambda^\varepsilon + 2\mu^\varepsilon$. We will show how the density $\rho = \rho^\circ + \rho^\varepsilon$ and the elastic modulus $E = \lambda + 2\mu = E^\circ + E^\varepsilon$ are determined.

We consider two force loading regimes corresponding to two basic dynamic processes

$$u_3^{\circ(1)} = \exp\left(-a_1 t - \frac{a_1}{c_1}x_3\right),\quad u_1^{\circ(1)} = u_2^{\circ(1)} = 0,\quad \Theta^{\circ(1)} = 0$$
$$u_3^{\circ(2)} = \exp\left(-a_2 t - \frac{a_2}{c_1}x_3\right),\quad u_1^{\circ(2)} = u_2^{\circ(2)} = 0,\quad \Theta^{\circ(2)} = 0$$
$$c_1 = (E^\circ/\rho^\circ)^{1/2},\quad a_1,\ a_2 > 0,\quad a_1 \neq a_2$$

Then, in conformity with the above algorithm, the dynamic processes in a weakly inhomogeneous medium are described by the problems

$$\begin{gathered}\rho^\circ V^{(k)\cdot\cdot} - E^\circ V^{(k)}_{,33} = 0,\quad V^{(k)}|_{t=0} = 0,\quad V^{(k)}_{,3}|_{x_3=0} = 0\\ V^{(k)}|_{x_3=0} = \chi^{(k)\cdot} + a_k\chi^{(k)} = d_k(t)\\ V^{(k)} = u_3^{\varepsilon(k)\cdot} + a_k u_3^{\varepsilon(k)},\quad u_3^{\varepsilon(k)}|_{x_3=0} = \chi^{(k)}(t),\quad k = 1, 2\end{gathered} \tag{3.4}$$

Exactly like problem (2.12), (2.13) considered in the general case, problems (3.4) refer to a type of non-hyperbolic Cauchy problem for the wave equation. They are solved considerably more simply in the one-dimensional case under consideration than in the three-dimensional case since they can be reduced to ordinary Cauchy problems by a simple renotation interchanging the places of the time and space variables. Using the D'Alembert formula, we obtain

$V^{(k)} = {}^1/_2\,[d_k\,(t - x_3/c_1) + d_k\,(t + x_3/c_1)]$ where the functions $d_k\,(t)$ are extended in an antisymmetric manner to the domain $t < 0$. It therefore results that $V^{(k)\cdot}|_{t=0} = d_k'\,(x_3/c_1)$.

Later following the general algorithm of the passage to differential equations in the thermoelastic characteristics, we obtain

$$\rho^\varepsilon a_k^2 \exp\left(-\frac{a_k}{c_1}x_3\right) + \frac{a_k}{c_1}\left[E^\varepsilon \exp\left(-\frac{a_k}{c_1}x_3\right)\right]_{,3} = -\rho^\circ d_k'\left(\frac{x_3}{c_1}\right), \quad k = 1, 2$$

We solve this system of equations taking the condition $E^\varepsilon|_{x_3=0} = 0$ into account. We find

$$E^\varepsilon(x_3) = \frac{c_1\rho^\circ}{(a_1 - a_2)}\int\limits_0^{x_3}\left[\frac{a_2}{a_1}d_1'\left(\frac{\eta}{c_1}\right)\exp\left(\frac{a_1}{c_1}\eta\right) - \frac{a_1}{a_2}d_2'\left(\frac{\eta}{c_1}\right)\exp\left(\frac{a_2}{c_1}\eta\right)\right]d\eta$$

$$\rho^\varepsilon(x_3) = \frac{\rho^\circ}{(a_1 - a_2)}\left[\frac{1}{a_2}\exp\left(\frac{a_2}{c_1}x_3\right)d_2'\left(\frac{x_3}{c_1}\right) - \frac{1}{a_1}\exp\left(\frac{a_1}{c_1}x_3\right)d_1'\left(\frac{x_3}{c_1}\right)\right] + \frac{E^\varepsilon}{c_1}$$

The dependence between χ^1, χ^2 and the thermoelastic characteristics is illustrated in a simple numerical example. To do this, we reduce (3.3) to dimensionless form by means of the formulas

$$\bar{u}_3^\varepsilon = \frac{u_3^\varepsilon}{b}, \quad \bar{\rho}^\varepsilon = \frac{\rho^\varepsilon}{\rho^\circ}, \quad \bar{E}^\varepsilon = \frac{E^\varepsilon}{E^\circ}, \quad \bar{t} = \sqrt{\frac{E^\circ}{\rho^\circ}}\,t$$

As the basic dynamic processes we take processes with the parameters $a_1 = {}^1/_2\,a_2 = 1$ and we consider the information obtained from experiment to have the form $\chi^{(1)} = 0.01\,(\exp(-\bar{t}) - 2\exp(-2\bar{t}) + \exp(-3\bar{t}))$, $\chi^{(2)} = 0$. The values $\bar{\rho} = 1 + \bar{\rho}^\varepsilon$, $\bar{E} = 1 + \bar{E}^\varepsilon$ are presented in Fig.1. Note that $\bar{\rho}^\varepsilon = 0$, $\bar{E}^\varepsilon = 0$ would be obtained for $\chi^{(1)} = \chi^{(2)} = 0$. This would mean that the external effect would not result in a change in the density and the modulus E.

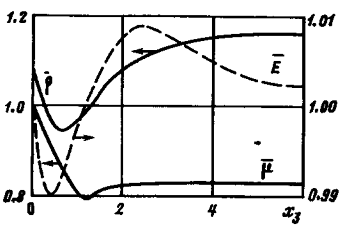

Fig.1

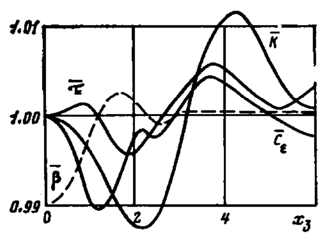

Fig.2

The algorithm to determine the Lamé coefficient $\mu = \mu^\circ + \mu^\varepsilon$ is analogous to the algorithm considered to determine E. Only now (3.2) is taken as basis in place of (3.3), and since the function ρ^ε has already been found, it is sufficient to conduct one test experiment with the basic dynamic process of the form

$$u_1^{\circ(3)} = \exp(-a_3 t - a_3 c_2^{-1} x_3), \quad u_2^{\circ(3)} = u_3^{\circ(3)} = 0, \quad \theta^{\circ(3)} = 0$$

$$c_2 = \sqrt{\mu^\circ/\rho^\circ}, \quad a_3 > 0$$

We have $u_1^\varepsilon|_{x_3=0} = \chi^{(3)}(t)$ as the information obtained from this experiment.

Performing the same computations as in the example considered for determining E^ε, ρ^ε, we obtain

$$\mu^\varepsilon(x_3) = -\frac{c_2}{a_3}\exp\left(\frac{a_3}{c_2}x_3\right)\int\limits_0^{x_3}\left[\rho^\circ d_3'\left(\frac{\eta}{c_2}\right) + a_3^2\rho^\varepsilon \exp\left(-\frac{a_3}{c_2}\eta\right)\right]d\eta$$

$$d_3 = \chi^{(3)\cdot} + a_3\chi^{(3)}$$

The results of a numerical computation (after an analogous reduction to dimensionless form) are presented in Fig.1 for $a_3 = 1$, $\chi^{(3)} = 0{,}01\,(2\exp(-\bar{t}) - 3\exp(-2\bar{t}) + \exp(-4\bar{t}))$. Here $\bar{\mu} = 1 + \mu^\varepsilon/\mu^\circ$.

The thermoelastic characteristics $C_\varepsilon^\varepsilon$, K^ε, τ will be sought by using experiments with temperature loading of the half-space boundary. We consider three basic dynamic processes for their determination

$$\theta^{\circ(n)} = \exp\left(\frac{K^\circ}{C_\varepsilon^\circ}a_n^2\bar{t} - a_n x_3\right), \quad n = 4, 5; \quad \theta^{\circ(6)} = x_3$$

$$u_i^{\circ(k)} = 0, \; k = 4, 5, 6, \; i = 1, 2, 3; \; a_4 = 1, \; a_5 = -2$$

and as information obtained from experiment we take

$$\theta^{\varepsilon(n)}\big|_{x_3=0} = \chi^{(n)}(t), \quad n = 4,\ 5$$

$$\theta^{\varepsilon(6)}\big|_{x_3=0} = \chi^{(6)}(t) = 0{,}1\,(\exp(-\bar{t}) - \exp(-2\bar{t}))$$

It is assumed that the temperature is already reduced to dimensionless form by the formula $\bar{\theta} = \theta/T_0$ but the bar is eliminated here and henceforth.

In this case, using (3.1) we obtain three equations to determine $C_\varepsilon^\varepsilon, K^\varepsilon, \tau$

$$C_\varepsilon{}^\varepsilon \frac{K^\circ}{C_\varepsilon{}^\circ} + K^\varepsilon_{,3} - K^\varepsilon - \tau_{,3}\frac{(K^\circ)^2}{C_\varepsilon{}^\circ} + \tau\frac{(K^\circ)^3}{C_\varepsilon{}^\circ} = 0$$

$$2C_\varepsilon{}^\varepsilon \frac{K^\circ}{C_\varepsilon{}^\circ} - K^\varepsilon_{,3} - 2K^\varepsilon + \tau_{,3}\frac{4(K^\circ)^2}{C_\varepsilon{}^\circ} + \tau\frac{8(K^\circ)^3}{C_\varepsilon{}^\circ} = 0$$

$$K^\varepsilon_{,3} = 0{,}1\,(\cos(2x_3) - \cos x_3)$$

The results of a numerical computation are presented in dimensionless form in Fig.2.

$$\bar{C}_\varepsilon{}^\varepsilon = \frac{C_\varepsilon{}^\varepsilon}{C_\varepsilon{}^\circ}, \quad \bar{K}^\varepsilon = \frac{K^\varepsilon}{K^\circ}, \quad \bar{\tau} = \frac{\tau}{C_\varepsilon{}^\circ}, \quad \bar{t} = \frac{K^\circ}{C_\varepsilon{}^\circ}t, \qquad \bar{C}_\varepsilon = 1 + \bar{C}_\varepsilon{}^\varepsilon, \quad \bar{K} = 1 + \bar{K}^\varepsilon$$

To determine the volume temperature expansion, coefficient β we consider a basic dynamic process of the form

$$\theta^{\circ(7)} = 0, \quad u_1^{\circ(7)} = u_2^{\circ(7)} = 0, \quad u_3^{\circ(7)} = \exp(-\bar{t} - x_3/c_1)$$

and we take as information

$$\theta^{\varepsilon(7)}\big|_{x_3=0} = \chi^{(7)}(t) = 0{,}2\left(\exp\left(-\frac{\bar{t}}{2}\right) + \exp\left(-\frac{3}{2}\bar{t}\right)\right)$$

The algorithm to determine β is exactly analogous to the algorithm to determine $C_\varepsilon{}^\varepsilon, K^\varepsilon, \tau$. Only in the last stage does the problem reduce not to a system of ordinary differential equations but to one algebraic equation since only the function β itself is in (3.1) and not its derivative. The function $\bar{\beta}$, reduced to dimensionless form by the formula $\bar{\beta} = \beta(C_\varepsilon{}^\circ T_0)^{-1}$, is shown in Fig.2.

Note that seven functions $\chi^{(n)}(t)(n = 1, \ldots, 7)$ of one variable are also required in addition to determine seven unknown functions (thermoelastic characteristics) of one variable (the spatial coordinates x_3). If $\chi^{(n)}(t) = 0$ $(n = 1, \ldots, 7)$, then this means that substantial changes in the thermoelastic properties of the material would not occur, and the initial model of a homogeneous isotropic medium can be used to describe the dynamic processes therein.

REFERENCES

1. LAVRENT'EV M.M., O Certain Ill-Posed Problems of Mathematical Physics. Izd. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1962.
2. PODSTRIGACH YA.S. and KOLYANO YU.M., Generalized Thermomechanics. Naukova Dumka, Kiev, 1976.
3. NOWACKI W., Dynamical Problems of Thermoelasticity. Mir, Moscow, 1970.
4. ROMANOV V.G., The inverse Lamb problem in the linear approximation, in: Numerical Methods in Seismic Research. 170-192, Vychisl. Tsentr Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1983.
5. NOWACKI W., Theory of Elasticity. Mir, Moscow, 1975.
6. TIKHONOV A.N. and SAMARSKII A.A., The Equations of Mathematical Physics. Nauka, Moscow, 1966.
7. BUKHGEIM A.L., Volterra Equations and Inverse Problems. Nauka, Novosibirsk, 1983.
8. GARIPOV R.M. and KARDAKOV V.V., The Cauchy problem for the wave equation with a non-spatial initial manifold. Dokl. Akad. Nauk SSSR, 213, 5, 1973.
9. LAVRENT'EV M.M., REZNITSKAYA K.G. and YAKHNO V.G., One-dimensional Inverse Problems of Mathematical Physics. Nauka, Novosibirsk, 1982.
10. REZNITSKAYA K.G., The relation between the solutions of the Cauchy problem for equations of different type and inverse problems. Mathematical Problems of Geophysics. Vychisl. Tsentr Sibirsk. Otdel. Akad. Nauk SSSR, 5, Pt.1, 1974.
11. COURANT R., Partial Differential Equations. Mir, Moscow, 1964.

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